OPERATOR-VALUED ANALYTIC FUNCTIONS ON SYMMETRIZED BIDISC

HARIPADA SAU

ABSTRACT. This paper is a continuation of work done in [10] to operator-valued analytic functions on the symmetrized bidisc

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\}.$$

For two Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$, let $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ denote the Banach space of bounded analytic $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$-valued functions on $\mathbb{G}$ equipped with the supremum norm. We give a concrete realization formula for functions in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$. We use this result to prove the operator-valued Nevanlinna-Pick interpolation in the space of bounded analytic functions on $\mathbb{G}$. We also prove the Toeplitz-corona problem on the symmetrized bidisc using the realization formula.

CONTENTS

1. INTRODUCTION

Let $H^\infty(\mathbb{D})$ denote the algebra of bounded holomorphic functions on the open unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{\mathbb{D}} |f(z)|$. Let $H^2(\mathbb{D})$ denote the Hilbert space of holomorphic functions on $\mathbb{D}$ whose Taylor coefficients at the origin are square-summable. It is well known that $H^\infty(\mathbb{D})$ is the multiplier algebra of $H^2(\mathbb{D})$. The following beautiful result which gives a representation of functions in $H^\infty(\mathbb{D})$ has played an important role both in complex analysis and operator theory.

Date: June 6, 2016.

MSC2010: 46E22, 47A13, 47A25, 47A56.

Key words and phrases: Symmetrized bidisc, Realization formula, Interpolation, Operator-valued kernel, Reproducing kernel Hilbert space, Toeplitz-corona problem.

The authors' research is supported by University Grants Commission Centre for Advanced Studies.

Theorem 1 (Theorem 6.5 in [2]). *A function f is in the closed unit ball of $H^\infty(\mathbb{D})$ if and only if there is a Hilbert space $\mathcal{H}$ and an isometry $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$ such that when V is written as*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1}$$

f can be represented as

$$f(\lambda) = A + \lambda B(I - \lambda D)^{-1}C.$$

This result is called a realization formula, because it gives a realization of the function f in a concrete form. The proof of this result uses basic operator theory, the theory of reproducing kernel Hilbert spaces and an elegant lurking isometry argument. Agler generalized this result successfully to the bidisc in [1]. He showed that a function f is in the closed unit ball of $H^\infty(\mathbb{D}^2)$, the algebra of bounded holomorphic functions on $\mathbb{D}^2$ equipped with the supremum norm, if and only if there are two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and an isometry $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that when V is written as in (1.1), f can be written as

$$f(\lambda) = A + \mathcal{E}_\lambda B(I - \mathcal{E}_\lambda D)^{-1}C,$$

where for $\lambda = (\lambda_1, \lambda_2)$ in the bidisc, $\mathcal{E}_\lambda$ denotes the operator $\lambda_1 I_{\mathcal{H}_1} \oplus \lambda_2 I_{\mathcal{H}_2}$. There are concrete generalizations of these realization theorems to the operator-valued case, see Chapter 11 of the classic book [2]. The symmetrized bidisc

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\}$$

is known to have similar operator theoretic aspects as the bidisc. Recently, a concrete realization formula for holomorphic functions acting on the symmetrized bidisc with the supremum norm no greater than one, has been described in [10]. This domain arose because of its connection with the μ-synthesis problem, posed by control engineers. The geometry, function theory and operator theory on this domain have been studied extensively in [4, 5, 6, 7, 9, 18] by several mathematicians over last two decades. A typical element of $\mathbb{G}$ will be denoted by (s, p), where s and p stand for 'sum' and 'product' respectively. There are several interesting characterizations of members of $\mathbb{G}$. Only one of them, which we mention next, will be used in this paper.

Lemma 2 (Theorem 2.1 in [7]). *For s, p in $\mathbb{C}$, the following statements are equivalent:*

(i) $(s, p) \in \mathbb{G}$;
(ii) $\left|\frac{2\alpha p - s}{2 - \alpha s}\right| < 1$ *for all* $\alpha \in \overline{\mathbb{D}}$.

A test case of the μ-synthesis problem is the spectral interpolation problem [8]. An attempt to solve a special case of the spectral interpolation problem led Agler and Young to define the following in [5].

Definition 3. A pair (S, P) of commuting bounded operators on a Hilbert space $\mathcal{H}$ is called a Γ-contraction if it has the closed symmetrized bidisc $\overline{\mathbb{G}}$ as a spectral set, i.e., if the Taylor joint spectrum of (S, P) is contained in $\overline{\mathbb{G}}$ and

$$\|f(S, P)\| \le \sup\{|f(s, p)| : (s, p) \in \overline{\mathbb{G}}\},$$

for all holomorphic function on a neighbourhood of $\overline{\mathbb{G}}$.

There are few characterizations of a Γ-contraction, we shall recall one of them, which will be used later in this paper.

Theorem 4 (Theorem 1.5 in [6]). *Let (S,P) be a pair of commuting bounded operators on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) (S,P) *is a* Γ*-contraction;*
(ii) $(2-\alpha S)(2-\alpha S)^* - (2\alpha^2 P - \alpha S)(2\alpha^2 P-\alpha S)^* \geq 0$ *for all* $\alpha \in \overline{\mathbb{D}}$.

Agler and Young used this characterization to show that for every Γ-contraction (S,P), the closed symmetrized bidisc $\overline{\mathbb{G}}$ is actually a complete spectral set, see [5]. In other words, if (S,P) is a Γ-contraction, then for every matrix-valued holomorphic function f in a neighbourhood of $\overline{\mathbb{G}}$,

$$\|f(S,P)\| \leq \sup\{\|f(s,p)\| : (s,p)\in \overline{\mathbb{G}}\}.$$

One purpose of this note is to extend the realization formula obtained in [10] to operator-valued analytic functions acting on the symmetrized bidisc. For two Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$, let $\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)$ denote the space of all bounded operators from $\mathcal{L}_1$ into $\mathcal{L}_2$. We consider the Banach space of $\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)$-valued bounded analytic functions on $\mathbb{G}$, and denote by $H^\infty\big(\mathbb{G},\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$. Theorem 14, the first main result of this paper, gives a realization formula for functions in the closed unit ball of $H^\infty\big(\mathbb{G},\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$.

Having proved this theorem, we turn our attention to the celebrated Nevanlinna-Pick interpolation problem. Given N points $\lambda_1,\lambda_2,\dots,\lambda_N$ in $\mathbb{D}$ and $w_1,w_2,\dots,w_N$ in the complex plane, the classical Nevanlinna-Pick problem asks when there is a function in the closed unit ball of $H^\infty(\mathbb{D})$ interpolating the given data. This problem was first investigated by Pick in [15] who proved that there is such a function if and only if the so called Pick matrix

$$\left(\frac{1-w_i\overline{w_j}}{1-\lambda_i\overline{\lambda_j}}\right)_{i,j=1}^{N}$$

is positive semi-definite. This problem has been studied extensively both by operator theoretic and function theoretic approaches. We use Theorem 14 to prove an operator version of Nevanlinna-Pick interpolation problem in $H^\infty\big(\mathbb{G},\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$ which has its own interest. More explicitly, we choose N points $(s_1,p_1),(s_2,p_2),\dots,(s_N,p_N)$ in $\mathbb{G}$ and N operators $W_1,W_2,\dots,W_N$ in $\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)$ and describe a necessary and sufficient condition when there is a function f in the closed unit ball of $H^\infty\big(\mathbb{G},\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$ interpolating the given data. We prove that such a function exists if and only if the following block operator matrix

$$\Big((I_{\mathcal{L}_2}-W_iW_j^*)\otimes K\big((s_i,p_i),(s_j,p_j)\big)\Big)_{i,j=1}^{N}$$

is positive semi-definite, for a family of $\mathcal{B}(\mathcal{L}_2)$-valued kernels K on $\mathbb{G}$, called the admissible kernels, which is defined in section 2.2. Theorem 15 not only proves the interpolation problem described above, but using the realization formula, it also describes all the solutions (interpolant) to the interpolation problem.

We also prove the Toeplitz-corona theorem for the symmetrized bidisc using the operator-valued version of the realization formula. The *corona problem* asks the following: Given functions $\varphi_1,\varphi_2,\dots,\varphi_N$ in $H^\infty(\mathbb{D})$ satisfying

$$|\varphi_1(z)|^2+|\varphi_2(z)|^2+\cdots+|\varphi_N(z)|^2 \geq \delta^2>0 \text{ for all } z\in\mathbb{D}, \tag{1.2}$$

for some positive real number δ, then do there exist functions $\psi_1,\psi_2,\dots,\psi_N$ in $H^\infty(\mathbb{D})$ such that

$$\psi_1\varphi_1+\psi_2\varphi_2+\cdots+\psi_N\varphi_N = 1? \tag{1.3}$$

Note that (1.3) is equivalent to the fact that the ideal generated by the functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ in $H^\infty(\mathbb{D})$ is full $H^\infty(\mathbb{D})$. Also note that condition (1.2) is a necessary condition for (1.3) to happen. The coronal problem was answered by L. Carleson in his seminal paper [11]. Later Hörmander [13] introduced a different approach to prove the coronal problem. His approach was dependent on solving an appropriate inhomogeneous $\bar{\partial}$-equations. See [23] and references therein for a beautiful discussion and various results in this direction. The corona problem admits the following generalization: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Hilbert spaces and $\Phi \in H^\infty\big(\mathbb{D}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ satisfying

$$\Phi(z)\Phi(z)^* \geq \delta^2 I_{\mathcal{L}_2} > 0, \tag{1.4}$$

does there exist a $\Psi \in H^\infty\big(\mathbb{D}, \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)\big)$ such that $\Phi(z)\Psi(z) = I_{\mathcal{L}_2}$ for all $z \in \mathbb{D}$? This generalization is called *operator corona problem.* This problem was posed by Sz.-Nagy in [20] in connection with problems in operator theory. Note that the case $\dim(\mathcal{L}_2) = 1$ and $\dim(\mathcal{L}_1) < \infty$ gives the classical corona problem. The case when $\dim(\mathcal{L}_2) = 1$ and $\dim(\mathcal{L}_1) = \infty$ was proved by Rosenblum [16] and Tolokonnikov [21] independently. But in the most general situation $\dim(\mathcal{L}_2) = \infty$ and $\dim(\mathcal{L}_1) = \infty$, the condition (1.4) does not imply the existence of a bounded analytic right inverse Ψ. In fact, it was shown by Treil [22] that the norm of the right inverse Ψ blows up, as $\dim(\mathcal{L}_2) \to \infty$. People were trying to find an operator theoretic proof of the corona problem. But only a weaker version of the problem, the so-called Toeplitz-corona problem was proved by Schubert [19].

Theorem 5 (Toeplitz-corona theorem)**.** *A function $\Phi \in H^\infty\big(\mathbb{D}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ is right invertible, i.e., there exists a function $\Psi \in H^\infty\big(\mathbb{D}, \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)\big)$ such that $\Phi(z)\Psi(z) = I_{\mathcal{L}_2}$ for all $z \in \mathbb{D}$ if and only if the Toeplitz operator T_Φ is right invertible, equivalently, T_Φ^* is left invertible. Moreover, if δ is a positive real number such that*

$$T_\Phi T_\Phi^* \geq \delta^2 I_{H^2(\mathcal{L}_2)},$$

then there exists a right inverse Ψ in $H^\infty\big(\mathbb{D}, \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)\big)$ with supremum norm no greater than $1/\delta$.

This result was proved initially for the case when $\dim(\mathcal{L}_2) = 1$ and $\dim(\mathcal{L}_1) < \infty$ by using the commutant lifting theorem. The general operator-valued case can be proved similarly. Theorem 5 does not quite give a solution to the corona problem, because in the case when $\dim(\mathcal{L}_2) = 1$ and $\dim(\mathcal{L}_1) = N$, there is no way to check whether condition (1.2) implies that the operator T_Φ has a right inverse. This theorem, however, is viewed as an independent result because the hypothesis, i.e., the operator T_Φ is right invertible, is relatively easier to check than condition (1.2), even when $\varphi_1, \varphi_2, \ldots, \varphi_N$ are polynomials. Also, unlike the corona theorem, this theorem gives the exact best bound for the norm of the function Ψ. The realization formula (Theorem 1) enables one to give an easier proof of Theorem 5, see Theorem 8.57 in [2]. In this paper, we use the realization formula (Theorem 14) for the symmetrized bidisc to prove the following Toeplitz-corona theorem for the symmetrized bidisc. Toeplitz-corona theorem on the bidisc was proved in [3].

Theorem 6. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be in $H^\infty(\mathbb{G})$ and $\delta > 0$. Then the following statements are equivalent:*

(i) *There exist functions $\psi_1, \psi_2, \ldots, \psi_N$ in $H^\infty(\mathbb{G})$ such that $\sum_{i=1}^N \varphi_i\psi_i = 1$ with $\sup_{\mathbb{G}} \sum_{i=1}^N |\psi_i(s,p)|^2 \leq \frac{1}{\delta}$;*

(ii) *The function* $\left(\sum_{i=1}^N \varphi_i(s,p)\overline{\varphi_i(t,q)} - \delta\right) k\big((s,p),(t,q)\big)$ *is positive semi-definite for every admissible kernel* k *on* $\mathbb{G}$*;*

(iii) *For every admissible kernel* k*, the multipliers* M_{φ_i} *on* H_k *satisfy the inequality* $\sum_{i=1}^N M_{\varphi_i} M_{\varphi_i}^* \geq \delta I$*, where* H_k *is the reproducing kernel Hilbert space associated with the kernel* k*.*

2. Preliminaries

In this section we shall briefly recall some basic properties of operator-valued kernels and introduce vector-valued Hardy space on the symmetrized bidisc. All the basic facts about vector-valued kernels which are discussed here, can also be found in [2]. We shall also define a special class of operator-valued kernels, called the admissible kernels.

2.1. **Operator-valued kernel.** Let $\mathcal{L}$ be a Hilbert space. Call an operator-valued function $K : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L})$ *self-adjoint* if $K(\lambda,\zeta) = K(\zeta,\lambda)^*$ for all $\lambda, \zeta \in \mathbb{G}$. A self-adjoint function $K : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L})$ is called a *positive semi-definite kernel* if for every finite choices of vectors $v_1, v_2, \ldots, v_N$ from $\mathcal{L}$,

$$\sum_{i,j=1}^N \langle K(\lambda_i,\lambda_j)v_j, v_i\rangle \geq 0.$$

In this paper, all the kernels are assumed to be analytic in the first variable and conjugate analytic in the second variable. For every operator-valued kernel K, we can define a Hilbert space H_K associated with it as

$$H_K = \bigvee\{g(\cdot) = \sum_{i=1}^N K((\cdot),(s_i,p_i))v_i : (s_i,p_i) \in \mathbb{G} \text{ and } v_i \in \mathcal{L}\},$$

with the inner product

$$\langle K((\cdot),(t,q))u, K((\cdot),(s,p))v\rangle_{H_K} = \langle K((s,p),(t,q))u, v\rangle_{\mathcal{L}}.$$

For two elements u, v in a Hilbert space $\mathcal{L}$, define the following rank one operator, sometimes called a *dyad*, $[u \otimes v] : \mathcal{L} \to \mathcal{L}$ by

$$[u \otimes v](l) = \langle l, v\rangle u, \text{ for all } l \in \mathcal{L}.$$

It is easy to check that $[u \otimes v] = [v \otimes u]^*$. Also for $A \in \mathcal{B}(\mathcal{L})$, it can be checked that

$$\begin{aligned} A[u \otimes v] &= [Au \otimes v], \\ [u \otimes v]A &= [u \otimes A^* v]. \end{aligned} \tag{2.1}$$

Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{L}$, then

$$\mathrm{tr}[u \otimes v] = \sum_n \langle [u \otimes v]e_n, e_n\rangle = \sum_n \langle e_n, v\rangle\langle u, e_n\rangle = \langle u, v\rangle.$$

With dyad operators defined, we now have a good source of operator-valued kernels.

Example 7. For a finite subset $Y = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of $\mathbb{G}$ and $v_1, v_2, \ldots, v_N \in \mathcal{L}$, we define the self-adjoint function $D : Y \times Y \to \mathcal{B}(\mathcal{L})$ by

$$D(\lambda_i, \lambda_j) = [v_i \otimes v_j]. \tag{2.2}$$

Note that for all $w_1, w_2, \ldots, w_N \in \mathcal{L}$ we have

$$\sum_{i,j=1}^{N} \langle [v_i \otimes v_j] w_j, w_i \rangle = \sum_{i,j=1}^{N} \langle w_j, v_j \rangle \langle v_i, w_i \rangle = \sum_{j=1}^{N} \langle w_j, v_j \rangle \sum_{i=1}^{N} \overline{\langle w_i, v_i \rangle} \geq 0.$$

Hence D is a positive semi-definite kernel on Y.

Let X be any set and $E : X \to \mathcal{B}(\mathcal{H}, \mathcal{L})$ be any function, where $\mathcal{H}$ and $\mathcal{L}$ are Hilbert spaces. It is easy to check that the function $E(\zeta)E(\lambda)^*$ defines a positive semi-definite kernel on X. The following well known result shows that the every operator-valued kernel arises this way.

Theorem 8 (Theorem 2.62 in [2]). *Let K be a $\mathcal{B}(\mathcal{L})$-valued kernel on a set X. Then there is a Hilbert space $\mathcal{H}$ and functions $F : X \to \mathcal{B}(\mathcal{L}, \mathcal{H})$ and $E : X \to \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that K can be written as*

$$K(\zeta, \lambda) = F(\zeta)^* F(\lambda) = E(\zeta)E(\lambda)^*.$$

Moreover, if K is analytic in ζ and conjugate analytic in λ, then F can be chosen to be conjugate analytic and E can be chosen to be analytic.

2.2. **Admissible kernel.** Let $\mathcal{L}$ be a Hilbert space and K be a $\mathcal{B}(\mathcal{L})$-valued kernel $\mathbb{G}$. Let M_s and M_p denote the multiplication by the coordinates operators on H_K, i.e.,

$$M_s(K\big((s,p),(t,q)\big)v) = sK\big((s,p),(t,q)\big)v \text{ and} \tag{2.3}$$

$$M_p(K\big((s,p),(t,q)\big)v) = pK\big((s,p),(t,q)\big)v, \tag{2.4}$$

for all $(s,p),(t,q) \in \mathbb{G}$ and $v \in \mathcal{L}$. By a well known result on multipliers on reproducing kernel Hilbert spaces, M_s and M_p are bounded operators on H_K of norm at most σ and ρ respectively if and only if the following holds:

$$\big(\sigma^2 - s\bar{t}\big) K\big((s,p),(t,q)\big) \geq 0 \text{ and } \big(\rho^2 - p\bar{q}\big) K\big((s,p),(t,q)\big) \geq 0.$$

When the pair (M_s, M_p) is defined on H_K, then for every $f \in H_K$ and $v \in \mathcal{L}$, we have

$$\langle M_s^* K((\cdot),(s,p))v, f(\cdot)\rangle = \langle K((\cdot),(s,p))v, M_s f\rangle = \langle \bar{s}K((\cdot),(s,p))v, f(\cdot)\rangle.$$

It is more convenient to write the function $K((\cdot),(s,p))$ simply by $K_{(s,p)}(\cdot)$. Therefore, with this notation, we have just obtained

$$M_s^* K_{(s,p)}(\cdot)v = \bar{s}K_{(s,p)}(\cdot)v.$$

Similar calculation yields

$$M_p^* K_{(s,p)}(\cdot)v = \bar{p}K_{(s,p)}(\cdot)v.$$

Definition 9. A kernel $K : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L})$ is called *admissible* if the pair (M_s, M_p) is a Γ-contraction on H_K.

Note that by Theorem 4, the pair (M_s, M_p) being a Γ-contraction on H_K is equivalent to the self-adjoint operator $(2-\alpha M_s)(2-\alpha M_s)^* - (2\alpha^2 M_p - \alpha M_s)(2\alpha^2 M_p - \alpha M_s)^*$ being positive semi-definite, for all $\alpha \in \overline{\mathbb{D}}$, which is equivalent to the following:

$$\Big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\Big) K\big((s,p),(t,q)\big) \geq 0 \text{ for all } \alpha \in \overline{\mathbb{D}}, \tag{2.5}$$

because

$$\begin{aligned} &\langle \Big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\Big) K\big((s,p),(t,q)\big)v, v'\rangle \\ = \;& \langle ((2-\alpha M_s)(2-\alpha M_t)^* - (2\alpha M_p - M_s)(2\alpha M_q - M_t)^*) K_{(t,q)}(\cdot)v, K_{(s,p)}(\cdot)v'\rangle. \end{aligned}$$

Also it follows from the definition that for a pair (S, P) to be a Γ-contraction, it is necessary that $\|S\| \le 2$ and $\|P\| \le 1$. Therefore we arrive at the following conclusion which we would not prove because it follows from the discussions above.

Lemma 10. *A kernel $K : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L})$ is admissible if and only if*

$$(4 - s\bar{t})\, K\big((s,p),(t,q)\big) \ge 0, \quad (1 - p\bar{q})\, K\big((s,p),(t,q)\big) \ge 0 \text{ and} \tag{2.6}$$

$$\Big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\Big)\, K\big((s,p),(t,q)\big) \ge 0 \text{ for all } \alpha \in \overline{\mathbb{D}}. \tag{2.7}$$

2.3. **Vector-valued Hardy space.** The vector-valued Hardy space of $\mathbb{G}$ provides us a natural example of an admissible kernel. That is what we define below.

Let $\mathcal{L}$ be a Hilbert space. Let us denote by $H^2(\mathcal{L})$ the class of functions holomorphic on $\mathbb{G}$ with values in $\mathcal{L}$ such that

$$\sup_{0<r<1} \int_{\mathbb{T}\times\mathbb{T}} \|f \circ \pi(re^{i\theta_1}, re^{i\theta_2})\|_{\mathcal{L}}^2 |J_\pi(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty,$$

where $J_\pi(z_1, z_2) = z_1 - z_2$, i.e., the complex Jacobian of the map

$$\pi(z_1, z_2) = (z_1 + z_2, z_1 z_2)$$

and $d\theta$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. The norm of a function f in $H^2(\mathcal{L})$ is defined to be

$$\|f\| = \|J\|^{-1}\Big(\sup_{0<r<1} \int_{\mathbb{T}\times\mathbb{T}} \|f \circ \pi(re^{i\theta_1}, re^{i\theta_2})\|_{\mathcal{L}}^2 |J_\pi(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2\Big)^{\frac{1}{2}}.$$

We multiply by the constant $\|J\|^{-1}$ to make sure that the constant functions $f(s,p) = l$, for all $(s,p) \in \mathbb{G}$ have norm $\|l\|$ and hence we can embed $\mathcal{L}$ into $H^2(\mathcal{L})$ isometrically.

When $\mathcal{L} = \mathbb{C}$, the corresponding $H^2(\mathcal{L})$ is the scalar-valued Hardy space of the symmetrized bidisc, denoted by $H^2(\mathbb{G})$. Note that the space $H^2(\mathcal{L})$ is isometrically isomorphic to $H^2(\mathbb{G}) \otimes \mathcal{L}$, via the unitary

$$f(\cdot) \otimes v \mapsto f(\cdot)v$$

for all $f \in H^2(\mathbb{G})$ and $v \in \mathcal{L}$. It is observed in [10] using a result obtained in [14], that $H^2(\mathbb{G})$ is a reproducing kernel Hilbert space with the kernel

$$K^S\big((s,p),(t,q)\big) = \frac{1}{(1-p\bar{q})^2 - (s - \bar{t}p)\overline{(t - \bar{s}q)}}. \tag{2.8}$$

The kernel K^S is the Szego kernel of the symmetrized bidisc, because the corresponding Hilbert space is the Hardy space on the symmmetrized bidisc. It is proved in [10] (Lemma 2.5) that the szego kernel is an admissible kernel, in other words, the following holds:

$$\Big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\Big)\, K^S\big((s,p),(t,q)\big) \ge 0 \text{ for all } \alpha \in \overline{\mathbb{D}}. \tag{2.9}$$

Now we define the operator-valued function $\mathbf{K}^S : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L})$ by

$$\mathbf{K}^S\big((s,p),(t,q)\big) = \frac{I_{\mathcal{L}}}{(1-p\bar{q})^2 - (s - \bar{t}p)\overline{(t - \bar{s}q)}}.$$

By what observed above, it follows that $H^2(\mathcal{L})$ is the reproducing kernel Hilbert space with the kernel $\mathbf{K}^S$ and that $\mathbf{K}^S$ is an admissible kernel. In other words, the pair of multiplication operator by the co-ordinate functions (M_s, M_p) is a Γ-contraction on $H^2(\mathcal{L})$.

3. The realization formula - operator case

It turns out that unlike the case of the unit disc and the unit bidisc, the Szego kernel does not help much to obtain the realization formula in the case of the symmetrized bidisc. Instead, a family of kernels is involved. We define it now. For each $\alpha \in \overline{\mathbb{D}}$, define the analytic map φ_α on $\mathbb{G}$ by

$$\varphi_\alpha(s,p) = \frac{2\alpha p - s}{2 - \alpha s} \text{ for all } (s,p) \in \mathbb{G}.$$

It follows from Lemma 2 that each φ_α maps the symmetrized bidisc into the open unit disc. Having noticed that, we define, for each $\alpha \in \overline{\mathbb{D}}$, the self-adjoint map $B_\alpha : \mathbb{G}\times\mathbb{G} \to \mathbb{C}$ by

$$B_\alpha\big((s,p),(t,q)\big) = \frac{1}{\Big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\Big)}.$$

Note that

$$B_\alpha\big((s,p),(t,q)\big) = \frac{1}{(2-\alpha s)\overline{(2-\alpha t)}} \cdot \frac{1}{1 - \varphi_\alpha(s,p)\overline{\varphi_\alpha(t,q)}}.$$

Therefore for each $\alpha \in \overline{\mathbb{D}}$, B_α is a positive semi-definite kernel on $\mathbb{G}$, since it is the Schur product of two positive semi-definite kernels.

Given a function $J : \mathbb{G}\times\mathbb{G} \to \mathcal{B}(\mathcal{L})$, define J_r for all $0 < r < 1$ by

$$J_r\big((s,p),(t,q)\big) = J\big((rs, r^2p),(rt, r^2q)\big)$$

for every (s,p) and (t,q) in $\mathbb{G}$. The following structure theorem plays a major role in finding the realization formula.

Theorem 11. *Let $J : \mathbb{G}\times\mathbb{G} \to \mathcal{B}(\mathcal{L})$ be a continuous self-adjoint function. Then*

$$J_r \oslash K : \big((s,p),(t,q)\big) \mapsto J\big((rs,r^2p),(rt,r^2q)\big) \otimes K\big((s,p),(t,q)\big) \tag{3.1}$$

is positive semi-definite for every $\mathcal{B}(\mathcal{L})$-valued admissible kernel K and every $0 < r < 1$ if and only if there is an $\alpha \in \overline{\mathbb{D}}$ and three positive semi-definite kernels Δ, Θ and Λ on $\mathbb{G}$ such that

$$\begin{aligned} J\big((s,p),(t,q)\big) &= (4 - s\bar t)\Theta\big((s,p),(t,q)\big) + (1 - p\bar q)\Lambda\big((s,p),(t,q)\big) \\ &\quad + \big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\big)\Delta\big((s,p),(t,q)\big). \end{aligned}$$

Proof. By definition of admissibility, that the condition is sufficient, is obvious. To prove the necessity direction, let $Y = \{(s_i,p_i) : 1 \le i \le N\}$ be a finite subset of $\mathbb{G}$. Consider the following subset of $N\times N$ self-adjoint matrices with entries in $\mathcal{B}(\mathcal{L})$,

$$\begin{aligned} \mathcal{W}_Y = \{&(4 - s_i\bar{s_j})\Theta\big((s_i,p_i),(s_j,p_j)\big) + (1 - p_i\bar{p_j})\Lambda\big((s_i,p_i),(s_j,p_j)\big) \\ &+ \big((2-\alpha s_i)\overline{(2-\alpha s_j)} - (2\alpha p_i - s_i)\overline{(2\alpha p_j - s_j)}\big)\Delta\big((s_i,p_i),(s_j,p_j)\big) \\ &: \alpha \in \overline{\mathbb{D}}, \Delta, \Theta \text{ and } \Lambda \text{ are positive semi-definite kernels on } Y \text{ and } 1 \le i,j \le N\}. \end{aligned}$$

Note that if we multiply a member of $\mathcal{W}_Y$ by a non-negative real number, then the element remains in $\mathcal{W}_Y$. Also $\mathcal{W}_Y$ is easily seen to be a convex set. Therefore $\mathcal{W}_Y$ is a wedge in the vector space of $N\times N$ self-adjoint matrices. The fact that for each $\alpha \in \overline{\mathbb{D}}$, B_α is a kernel on $\mathbb{G}$ implies the kernel $\mathbf{1}\big((s,p),(t,q)\big) = I_{\mathcal{L}}$ is in $\mathcal{W}_Y$. Since the Schur product of two positive semi-definite kernels is again positive semi-definite, Schur product of any element in $\mathcal{W}_Y$ with any $\mathcal{B}(\mathcal{L})$-valued $N\times N$ positive semi-definite matrix is again in $\mathcal{W}_Y$, which in turn implies that any $\mathcal{B}(\mathcal{L})$-valued $N\times N$ positive semi-definite

matrix is in $\mathcal{W}_Y$, being the Schur product of itself and $\mathbf{1}$. In particular, the kernel D defined in (2.2) is in $\mathcal{W}_Y$. Note that $\mathcal{W}_Y$ can be viewed as a subset of $\mathcal{B}(\mathcal{L}^N)$, which is the dual of $\mathcal{B}_1(\mathcal{L}^N)$, the ideal of trace class operators acting on $\mathcal{L}^N = \mathcal{L}\oplus\mathcal{L}\oplus\cdots\oplus\mathcal{L}$ (N-times). We show that $\mathcal{W}_Y$ is weak-star closed. It was proved in [10] that $\mathcal{W}_Y$ is closed when the kernels in question are scalar-valued. We shall use that here. Let

$$\begin{aligned} K_{\alpha_n} &= (4-s_i\bar{s_j})\Theta_n\big((s_i,p_i),(s_j,p_j)\big) + (1-p_i\bar{p_j})\Lambda_n\big((s_i,p_i),(s_j,p_j)\big) \\ &+ \big((2-\alpha_n s_i)\overline{(2-\alpha_n s_j)} - (2\alpha_n p_i - s_i)\overline{(2\alpha_n p_j - s_j)}\big)\Delta_n\big((s_i,p_i),(s_j,p_j)\big) \end{aligned}$$

be a sequence in $\mathcal{W}_Y$ converging to $K = (K_{ij})_{1\le i,j\le N}$ with respect to the weak-star topology. That means, for every $X = (X_{kl}) \in \mathcal{B}_1(\mathcal{L}^N)$, the sequence of scalars $\mathrm{tr}(K_{\alpha_n}X)$ will converge to $\mathrm{tr}(KX)$. Let u and v be two vectors in $\mathcal{L}$. For $1\le i,j\le N$, choosing X^{ij} to be the operator in the trace class family with $[u\otimes v]$ at the (ji)-th entry and zero at all other entries, we get

$$\begin{aligned} (3.2)\qquad & (4-s_i\bar{s_j})\langle\Theta_n\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle + (1-p_i\bar{p_j})\langle\Lambda_n\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle \\ & +\big((2-\alpha_n s_i)\overline{(2-\alpha_n s_j)} - (2\alpha_n p_i - s_i)\overline{(2\alpha_n p_j - s_j)}\big)\langle\Delta_n\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle \\ & \to \langle K_{ij}u,v\rangle \text{ as } n\to\infty. \end{aligned}$$

Note that if Δ is a $\mathcal{B}(\mathcal{L})$-valued kernel on Y, then for every $u,v\in\mathcal{L}$,

$$\Delta^{u,v}\big((s_i,p_i),(s_j,p_j)\big) = \langle\Delta\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle$$

defines a scalar-valued kernel on Y. Therefore rephrasing (3.2) we get for every $u,v\in\mathcal{L}$,

$$\begin{aligned} &(4-s_i\bar{s_j})\Theta_n^{u,v}\big((s_i,p_i),(s_j,p_j)\big) + (1-p_i\bar{p_j})\Lambda_n^{u,v}\big((s_i,p_i),(s_j,p_j)\big) \\ &+\big((2-\alpha_n s_i)\overline{(2-\alpha_n s_j)} - (2\alpha_n p_i - s_i)\overline{(2\alpha_n p_j - s_j)}\big)\Delta_n^{u,v}\big((s_i,p_i),(s_j,p_j)\big) \to \langle K_{ij}u,v\rangle \end{aligned}$$

as $n\to\infty$, for every $1\le i,j\le N$. Now we are in the scalar-valued kernel situation. Therefore as in the proof of Theorem 3.4 in [10], we shall have an $\alpha\in\overline{\mathbb{D}}$ and scalar-valued kernels $\Theta^{u,v}, \Lambda^{u,v}$ and $\Delta^{u,v}$ on Y such that

$$\begin{aligned} &(4-s_i\bar{s_j})\Theta^{u,v}\big((s_i,p_i),(s_j,p_j)\big) + (1-p_i\bar{p_j})\Lambda^{u,v}\big((s_i,p_i),(s_j,p_j)\big) \\ &+\big((2-\alpha s_i)\overline{(2-\alpha s_j)} - (2\alpha p_i - s_i)\overline{(2\alpha p_j - s_j)}\big)\Delta^{u,v}\big((s_i,p_i),(s_j,p_j)\big) = \langle K_{ij}u,v\rangle \end{aligned}$$

for every $1\le i,j\le N$. Define the operator-valued kernel Θ on Y by

$$\langle\Theta\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle = \Theta^{u,v}\big((s_i,p_i),(s_j,p_j)\big)$$

for every $1\le i,j\le N$ and $u,v\in\mathcal{L}$. Similarly, we define Λ and Δ on Y to obtain

$$\begin{aligned} &(4-s_i\bar{s_j})\langle\Theta\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle + (1-p_i\bar{p_j})\langle\Lambda\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle \\ &+\big((2-\alpha s_i)\overline{(2-\alpha s_j)} - (2\alpha p_i - s_i)\overline{(2\alpha p_j - s_j)}\big)\langle\Delta\big((s_i,p_i),(s_j,p_j)\big)u,v\rangle = \langle K_{ij}u,v\rangle, \end{aligned}$$

which proves that $\mathcal{W}_Y$ is closed with respect to the weak-star topology and hence with respect to the operator norm topology also.

We now show that the restriction j of J to $Y\times Y$ is in $\mathcal{W}_Y$. Suppose on the contrary that j is not in $\mathcal{W}_Y$. Then, applying part (b) of Theorem 3.4 in [17], we get a weak-star continuous linear functional L on $\mathcal{B}(\mathcal{L}^N)$, whose real part is non-negative on $\mathcal{W}_Y$ and strictly negative on j. We replace this linear functional by its real part, i.e., $\frac{1}{2}(L(T)+\overline{L(T)})$ and denote it by L itself. Since L is weak-star continuous, by Theorem 1.3 in Chapter V of [12] we get L to be of the form

$$L(T) = \mathrm{tr}(TK)$$

for some $N \times N$ self-adjoint matrix K with entries in the ideal of trace class operators. Let us define K^t by $K^t(\lambda_i, \lambda_j) = K(\lambda_j, \lambda_i)^t$. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{L}$. For $u = \sum c_m e_m$ and $v = \sum d_n e_n$ in $\mathcal{L}$, we make a note of the following fact about K^t which will be used later in the proof.

$$\begin{aligned}\langle K^t(\lambda_i, \lambda_j)u, v\rangle &= \sum_{m,n} c_m \bar{d_n} \langle K(\lambda_j, \lambda_i)^t e_m, e_n\rangle \\ &= \sum_{m,n} c_m \bar{d_n} \langle K(\lambda_j, \lambda_i) e_n, e_m\rangle = \langle K(\lambda_j, \lambda_i)\bar{v}, \bar{u}\rangle,\end{aligned}$$

where $\bar{u} = \sum \bar{c_m} e_m$ and $\bar{v} = \sum \bar{d_n} e_n$.

We first prove that K^t is a $\mathcal{B}(\mathcal{L})$-valued positive semi-definite kernel on Y, i.e., we prove

$$\sum_{i,j=1}^{N} \langle K^t(\lambda_i, \lambda_j) v_j, v_i\rangle \geq 0, \tag{3.3}$$

where $v_1, v_2, \dots, v_N$ are arbitrary vectors in $\mathcal{L}$. The following shows that (3.3) is the action of L on the kernel $D_{ij} = [\bar{v_i} \otimes \bar{v_j}]$ and hence we are done.

$$\begin{aligned}0 \leq L(D) = \operatorname{tr}(DK) = \sum_{i,j=1}^{N} \operatorname{tr}(D_{ij} K_{ji}) &= \sum_{i,j=1}^{N} \operatorname{tr}([\bar{v_i} \otimes K_{ji}^* \bar{v_j}]) = \sum_{i,j=1}^{N} \langle \bar{v_i}, K_{ji}^* \bar{v_j}\rangle \\ &= \sum_{i,j=1}^{N} \langle K_{ji} \bar{v_i}, \bar{v_j}\rangle = \sum_{i,j=1}^{N} \langle K^t(\lambda_i, \lambda_j) v_j, v_i\rangle.\end{aligned}$$

Now that we know K^t is a positive semi-definite kernel, let us check whether K^t is admissible. Therefore we have to check whether K^t satisfies conditions (2.6) and (2.7). This is an easy task of choosing the kernels appropriately. For example, choosing $\Theta(\lambda_i, \lambda_j) = [\bar{v_i} \otimes \bar{v_j}]$ for a given finite subset $\{v_1, v_2, \dots, v_N\}$ of $\mathcal{L}$ and $\Lambda = 0 = \Delta$ we get the corresponding element $\bar{c_i} c_j (4 - s_i \bar{s_j})$ of $\mathcal{W}_Y$. The fact that L is non-negative on $\mathcal{W}_Y$ proves that K^t satisfies the first condition of (2.6). Similarly, one can check that K^t satisfies the other two conditions of admissibility.

Therefore by assumption, the $\mathcal{B}(\mathcal{L} \otimes \mathcal{L})$-valued function $j_r \oslash K^t$ on $Y \times Y$ is positive semi-definite, which means that for every choice of vectors $u_i \in \mathcal{L} \otimes \mathcal{L}$, $1 \leq i \leq N$ we have

$$\sum_{i,j=1}^{N} \langle j_r \oslash K^t(\lambda_i, \lambda_j) u_j, u_i\rangle \geq 0. \tag{3.4}$$

For a finite subset $\mathcal{F} = \{1, 2, \dots, R\}$ of $\mathbb{N}$, choose $u_i = \sum_{m=1}^{R} e_m \otimes e_m$, for each i. Note that for this choice of u_i, (3.4) is same as

$$\sum_{i,j=1}^{N} \sum_{m,n=1}^{R} \langle j_r(\lambda_i, \lambda_j) e_m, e_n\rangle \langle K^t(\lambda_i, \lambda_j) e_m, e_n\rangle \geq 0. \tag{3.5}$$

Note that for every $0 < r < 1$,

$$\begin{aligned} L(j_r) &= \sum_{i,j=1}^{N} \mathrm{tr}(j_r(\lambda_i,\lambda_j)K(\lambda_j,\lambda_i)) \\ &= \sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\langle j_r(\lambda_i,\lambda_j)K(\lambda_j,\lambda_i)e_n, e_n\rangle \\ &= \sum_{i,j=1}^{N}\sum_{m,n=1}^{\infty}\langle j_r(\lambda_i,\lambda_j)e_m, e_n\rangle\langle K(\lambda_j,\lambda_i)e_n, e_m\rangle \\ &= \sum_{i,j=1}^{N}\sum_{m,n=1}^{\infty}\langle j_r(\lambda_i,\lambda_j)e_m, e_n\rangle\langle K^t(\lambda_i,\lambda_j)e_m, e_n\rangle \geq 0. \end{aligned}$$

The last inequality follows from (3.5). Now we use the continuity continuity of J and L to get $L(j) \geq 0$, which is contradiction to the assumption that J not in $\mathcal{W}_Y$. Now a standard application of Kurosh's theorem (Theorem 2.56 in [2]) completes the proof. $\square$

Remark 12. Note that Theorem 11 will still remain true, if one assumes that

$$J \oslash K : \big((s,p),(t,q)\big) \mapsto J\big((s,p),(t,q)\big) \otimes K\big((s,p),(t,q)\big)$$

is positive semi-definite for every admissible kernel K. Also, note that we do not need continuity of J, if we work with this assumption. One more advantage with this assumption is that the theorem can be extended to a function J which is defined on any arbitrary subset of the symmetrized bidisc, even on finite subsets. Instead of this, we made a slightly weaker assumption, namely (3.1), to facilitate a direct application. It will become clear from the proof of Lemma 13.

Lemma 13. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Hilbert spaces. If f is a function in $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$, then there exist an $\alpha \in \overline{\mathbb{D}}$ and positive semi-definite $\mathcal{B}(\mathcal{L}_2)$-valued kernels Θ, Λ and Δ on $\mathbb{G}$ such that*

$$\begin{aligned} I_{\mathcal{L}_2} - f(s,p)f(t,q)^* &= (4 - s\bar{t})\Theta\big((s,p),(t,q)\big) + (1 - p\bar{q})\Lambda\big((s,p),(t,q)\big) \\ &+ \big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\big)\Delta\big((s,p),(t,q)\big). \end{aligned}$$

Proof. We shall use Theorem 11 to prove this result. We shall show that if K is a $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel on $\mathbb{G}$, then

$$\big(I_{\mathcal{L}_2} - f_r(s,p)f_r(t,q)^*\big) \oslash K\big((s,p),(t,q)\big)$$

is positive semi-definite kernel, where $f_r(s,p) = f(rs, r^2p)$, for all $(s,p) \in \mathbb{G}$. Therefore we have to show that the following matrix

$$\Big(\big(I_{\mathcal{L}_2} - f_r(s_i,p_i)f_r(s_j,p_j)^*\big) \otimes K\big((s_i,p_i),(s_j,p_j)\big)\Big)_{i,j=1}^{N}$$

is positive semi-definite for every choice of the points $\{(s_j,p_j) : 1 \leq j \leq N\}$. Note that each entry of the matrix is an operator in $\mathcal{B}(\mathcal{L}_2 \otimes \mathcal{L}_2)$. Write $w_i = \sum_m u_{im} \otimes v_{im}$, we have to show that

$$\sum_{i,j}\sum_{m,n}\langle\big(I_{\mathcal{L}_2} - f_r(s_i,p_i)f_r(s_j,p_j)^*\big)u_{jn}, u_{im}\rangle_{\mathcal{L}_2}\langle K\big((s_i,p_i),(s_j,p_j)\big)v_{jn}, v_{im}\rangle_{\mathcal{L}_2} \geq 0.$$

The fact that K is admissible gives us the pair (M_s, M_p) is a Γ-contraction on the kernel's associated Hilbert space H_K. Note that if f is in $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$, then the function defined by $\check{f}(s,p) = f(\bar{s}, \bar{p})^*$ is in $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1))$ and since $\mathbb{G}$ is invariant under complex conjugate, it follows that f and $\check{f}$ have the same supremum norm. Note that the pair (M_s, M_p) need not have full H^∞ functional calculus, so for f in $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ and $0 < r < 1$, we shall consider the function $\check{f}_r(s,p) = f(r\bar{s}, r^2\bar{p})^*$. Since (M_s, M_p) is a Γ-contraction, $\check{f}_r(M_s^*, M_p^*)$ is a contraction, for all $0 < r < 1$, where by definition of M_s and M_p, we have

$$\begin{aligned} \check{f}_r(M_s^*, M_p^*) \ &: \ H_K \otimes \mathcal{L}_2 \to H_K \otimes \mathcal{L}_1 \\ & \quad\ K_{(s,p)} u \otimes v \mapsto K_{(s,p)} u \otimes f_r(s,p)^* v. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 \ &\leq \ \langle \big(I - \check{f}_r(M_s^*, M_p^*)^* \check{f}_r(M_s^*, M_p^*)\big) \sum_{j,n} K_{(s_j,p_j)} v_{jn} \otimes u_{jn}, \sum_{i,m} K_{(s_i,p_i)} v_{im} \otimes u_{im} \rangle \\ &= \ \sum_{i,j,m,n} \langle \big(1 - f_r(s_i,p_i) f_r(s_j,p_j)^*\big) u_{jn}, u_{im} \rangle \langle k\big((s_i,p_i),(s_j,p_j)\big) v_{jn}, v_{im} \rangle. \end{aligned}$$

So, by Theorem 11, we are done. $\square$

For $(s,p) \in \mathbb{G}$, and two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, define the following strict contraction on $\mathcal{H} \oplus \mathcal{K}$,

$$\mathcal{E}(s,p) = \begin{pmatrix} \frac{s}{2} I_{\mathcal{H}} & 0 \\ 0 & p I_{\mathcal{K}} \end{pmatrix}.$$

We call $\mathcal{E}(s,p)$ the evaluation operator. Now, we are ready to state and prove the main result of this section.

Theorem 14. *A function f is in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ if and only if there is an $\alpha \in \overline{\mathbb{D}}$, Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a unitary $V : \mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K} \to \mathcal{L}_2 \oplus \mathcal{H} \oplus \mathcal{K}$ such that when V is written as the matrix*

$$\begin{pmatrix} P & Q & R \\ S & T & U \\ W & X & Y \end{pmatrix} \tag{3.6}$$

f can be realized as

$$f(s,p) = A(s,p) + B(s,p)\mathcal{E}(s,p)(I - D(s,p)\mathcal{E}(s,p))^{-1} C(s,p), \tag{3.7}$$

where A, B, C and D are the following operator-valued analytic functions on $\mathbb{G}$

$$\begin{aligned} A(s,p) &= P + (2\alpha p - s) R\big((2 - \alpha s) - (2\alpha p - s) Y\big)^{-1} W \\ B(s,p) &= Q + (2\alpha p - s) R\big((2 - \alpha s) - (2\alpha p - s) Y\big)^{-1} X \\ C(s,p) &= S + (2\alpha p - s) U\big((2 - \alpha s) - (2\alpha p - s) Y\big)^{-1} W \\ D(s,p) &= T + (2\alpha p - s) U\big((2 - \alpha s) - (2\alpha p - s) Y\big)^{-1} X \end{aligned} \tag{3.8}$$

such that

$$T_{V,\alpha}(s,p) = \begin{pmatrix} A(s,p) & B(s,p) \\ C(s,p) & D(s,p) \end{pmatrix} \tag{3.9}$$

is a contraction for every $(s,p) \in \mathbb{G}$.

Proof. We first prove that the condition is sufficient. So suppose f is a function on $\mathbb{G}$ for which there exists a unitary V with the matrix form as in (3.6) such that f has the form as (3.7). Analyticity of f is clear. To show that it has its norm no greater than one, let us define the operator-valued analytic function G on $\mathbb{G}$ by $G(s,p) = (I - D(s,p)\mathcal{E}(s,p))^{-1}C(s,p)$. Then by (3.7) we have

$$\begin{pmatrix} A(s,p) & B(s,p) \\ C(s,p) & D(s,p) \end{pmatrix} \begin{pmatrix} I_{\mathcal{L}_1} \\ \mathcal{E}(s,p)G(s,p) \end{pmatrix} = \begin{pmatrix} f(s,p) \\ G(s,p) \end{pmatrix}.$$

The fact that for all $(s,p) \in \mathbb{G}$, the 2×2 block operator matrix $T_{V,\alpha}(s,p)$ is a contraction gives

$$\|f(s,p)\|^2 + \|G(s,p)\|^2 \leq 1 + \|\mathcal{E}(s,p)G(s,p)\|^2,$$

from which follows that f is in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$.

Conversely, let f be a function in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$. By Lemma 13, there exist an $\alpha \in \overline{\mathbb{D}}$ and positive semi-definite $\mathcal{B}(\mathcal{L}_2)$-valued kernels Θ, Λ and Δ on $\mathbb{G}$ such that

$$\begin{aligned} (3.10)\quad 1 - f(s,p)f(t,q)^* &= (4 - s\bar{t})\Theta\big((s,p),(t,q)\big) + (1 - p\bar{q})\Lambda\big((s,p),(t,q)\big) \\ &+ \big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\big)\Delta\big((s,p),(t,q)\big). \end{aligned}$$

Let $H_1 : \mathbb{G} \to \mathcal{B}(\mathcal{H}_1, \mathcal{L}_2)$, $H_2 : \mathbb{G} \to \mathcal{B}(\mathcal{H}_2, \mathcal{L}_2)$ and $G : \mathbb{G} \to \mathcal{B}(\mathcal{K}, \mathcal{L}_2)$ be analytic functions such that

$$\Theta\big((s,p),(t,q)\big) = H_1(s,p)H_1(t,q)^*, \Lambda(\big((s,p),(t,q)\big) = H_2(s,p)H_2(t,q)^* \text{ and}$$
$$\Delta\big((s,p),(t,q)\big) = G(s,p)G(t,q)^*.$$

Note that by Theorem 8 such analytic functions exist. In view of this, equality (5.1) becomes

$$\begin{aligned} & I_{\mathcal{L}_2} + s\bar{t}H_1(s,p)H_1(t,q)^* + p\bar{q}H_2(s,p)H_2(t,q)^* + (2\alpha p - s)\overline{(2\alpha q - t)}G(s,p)G(t,q)^* \\ =\ & f(s,p)f(t,q)^* + 4H_1(s,p)H_1(t,q)^* + H_2(s,p)H_2(t,q)^* + (2-\alpha s)\overline{(2-\alpha t)}G(s,p)G(t,q)^*. \end{aligned}$$

Define the operator-valued function $H(s,p)$ on $\mathbb{G}$ by $H(s,p) = \big(2H_1(s,p), H_2(s,p)\big)$. Note that for each (s,p) in $\mathbb{G}$, $H(s,p)$ is an operator from the space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{L}_2$. The preceding equality tells us that there exists an isometry V_1 from the span of

$$\{I_{\mathcal{L}_2}v \oplus \mathcal{E}(t,q)^*H(t,q)^*v \oplus \overline{(2\alpha q - t)}G(t,q)^*v : v \in \mathcal{L}_2, (t,q) \in \mathbb{G}\} \subset \mathcal{L}_2 \oplus \mathcal{H} \oplus \mathcal{K}$$

onto the span of

$$\{f(t,q)^*v \oplus H(t,q)^*v \oplus \overline{(2 - \alpha t)}G(t,q)^*v : v \in \mathcal{L}_2, (t,q) \in \mathbb{G}\} \subset \mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K}$$

such that for all $v \in \mathcal{L}_2$,

$$(3.11)\qquad \begin{pmatrix} I_{\mathcal{L}_2} \\ \mathcal{E}(t,q)^*H(t,q)^* \\ \overline{(2\alpha q - t)}G(t,q)^* \end{pmatrix} v \xrightarrow{V_1} \begin{pmatrix} f(t,q)^* \\ H(t,q)^* \\ \overline{(2-\alpha t)}G(t,q)^* \end{pmatrix} v.$$

Now a standard technique of adding an infinite dimensional summand to both $\mathcal{H}$ and $\mathcal{K}$, if necessary, we can extend V_1 unitarily from whole of $\mathcal{L}_2 \oplus \mathcal{H} \oplus \mathcal{K}$ onto $\mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K}$. Let the following be the 3×3 block operator matrix representation for V_1

$$(3.12)\qquad \begin{pmatrix} P_1 & Q_1 & R_1 \\ S_1 & T_1 & U_1 \\ W_1 & X_1 & Y_1 \end{pmatrix}.$$

From (3.11) we get

$$P_1 + Q_1\mathcal{E}(t,q)^*H(t,q)^* + \overline{(2\alpha q - t)}R_1G(t,q)^* = f(t,q)^* \tag{3.13}$$
$$S_1 + T_1\mathcal{E}(t,q)^*H(t,q)^* + \overline{(2\alpha q - t)}U_1G(t,q)^* = H(t,q)^* \tag{3.14}$$
$$W_1 + X_1\mathcal{E}(t,q)^*H(t,q)^* + \overline{(2\alpha q - t)}Y_1G(t,q)^* = \overline{(2-\alpha t)}G(t,q)^*. \tag{3.15}$$

Eliminating $H(t,q)^*$ and $G(t,q)^*$ we get

$$f(t,q)^* = A_1(t,q) + B_1(t,q)\mathcal{E}(t,q)^*(I - D_1(t,q)\mathcal{E}(t,q)^*)^{-1}C_1(t,q), \tag{3.16}$$

where

$$\begin{aligned} A_1(t,q) &= P_1 + \overline{(2\alpha q - t)}R_1\big(\overline{(2-\alpha t)} - \overline{(2\alpha q - t)}Y_1\big)^{-1}W_1 \\ B_1(t,q) &= Q_1 + \overline{(2\alpha p - s)}R_1\big(\overline{(2-\alpha t)} - \overline{(2\alpha q - t)}Y_1\big)^{-1}X_1 \\ C_1(t,q) &= S_1 + \overline{(2\alpha p - s)}U_1\big(\overline{(2-\alpha t)} - \overline{(2\alpha q - t)}Y_1\big)^{-1}W_1 \quad \text{and} \\ D_1(t,q) &= T_1 + \overline{(2\alpha p - s)}U_1\big(\overline{(2-\alpha t)} - \overline{(2\alpha q - t)}Y_1\big)^{-1}X_1. \end{aligned} \tag{3.17}$$

Taking adjoint of (3.16) we get the desired form of f

$$f(s,p) = A(s,p) + B(s,p)\mathcal{E}(s,p)(I - D(s,p)\mathcal{E}(s,p))^{-1}C(s,p), \text{ for all } (s,p)\in\mathbb{G},$$

where $A(s,p), B(s,p), C(s,p)$ and $D(s,p)$ are the adjoint of $A_1(s,p), C_1(s,p), B_1(s,p)$ and $D_1(s,p)$ respectively. And note that the unitary V that works for us is the adjoint of V_1. Writing V in the 3×3 matrix form

$$V = \begin{pmatrix} P & Q & R \\ S & T & U \\ W & X & Y \end{pmatrix} = \begin{pmatrix} P_1^* & S_1^* & W_1^* \\ Q_1^* & T_1^* & X_1^* \\ R_1^* & U_1^* & Y_1^* \end{pmatrix} = V_1^*,$$

it is easy to see from (3.17) that the functions A, B, C and D have the expression as stated in the theorem. The proof of the fact that $T_{V,\alpha}(s,p)$ is a contraction for all $(s,p)\in\mathbb{G}$ is the same as the proof of Lemma 3.6 in [10]. So we omit it. $\square$

4. The Nevanlinna-Pick interpolation

In this section we use the realization formula obtained in the previous section, to solve the Nevanlinna-Pick interpolation problem in $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Hilbert spaces. We show that the criterion for the existence of an interpolant involves not only the Szego kernel but a whole family of kernels, the admissible ones.

Theorem 15. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces and $W_1, W_2, \dots, W_N \in \mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)$. Let $(s_1,p_1),(s_2,p_2),\dots,(s_N,p_N)$ be N distinct points in $\mathbb{G}$. Then there exists a function f in the closed unit ball of $H^\infty\big(\mathbb{G},\mathcal{B}(\mathcal{L}_1,\mathcal{L}_2)\big)$ interpolating each (s_i,p_i) to W_i if and only if*

$$\Big((I_{\mathcal{L}_2} - W_iW_j^*)\otimes K\big((s_i,p_i),(s_j,p_j)\big)\Big)_{i,j=1}^N \ge 0 \tag{4.1}$$

for every $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel K on $\mathbb{G}$.

Proof. That the condition is necessary follows from Lemma 13. It is the converse part that depends heavily on the realization formula. So suppose (4.1) holds. We have to

find an interpolant with its norm no greater than one. By Theorem 11 and the remark following it, there is an $\alpha \in \overline{\mathbb{D}}$ and three operator-valued kernels Δ, Θ and Λ such that

$$
\begin{aligned}
(4.2)\ I_{\mathcal{L}_2} - W_i W_j^* &= (4 - s_i\bar{s_j})\Theta\big((s_i,p_i),(s_j,p_j)\big) + (1 - p_i\bar{p_j})\Lambda\big((s_i,p_i),(s_j,p_j)\big) \\
&+ \big((2-\alpha s_i)\overline{(2-\alpha s_j)} - (2\alpha p_i - s_i)\overline{(2\alpha p_j - s_j)}\big)\Delta\big((s_i,p_i),(s_j,p_j)\big).
\end{aligned}
$$

Factor the kernels Θ,Λ and Δ, according to Theorem 8 as

$$
\begin{aligned}
&\Theta\big((s,p),(t,q)\big) = H_1(s,p)H_1(t,q)^*, \Lambda\big((s,p),(t,q)\big) = H_2(s,p)H_2(t,q)^* \text{ and} \\
&\Delta\big((s,p),(t,q)\big) = G(s,p)G(t,q)^*,
\end{aligned}
$$

where $H_1 : \mathbb{G} \to \mathcal{B}(\mathcal{H}_1, \mathcal{L}_2)$, $H_2 : \mathbb{G} \to \mathcal{B}(\mathcal{H}_2, \mathcal{L}_2)$ and $G : \mathbb{G} \to \mathcal{B}(\mathcal{K}, \mathcal{L}_2)$ are analytic functions and $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{K}$ are some auxiliary Hilbert spaces. Then we have by a rearrangement of terms in (4.2)

$$
\begin{aligned}
I_{\mathcal{L}_2} + s_i\bar{s_j}H_1(s_i,p_i)H_1(s_j,p_j)^* &+ p_i\bar{p_j}H_2(s_i,p_i)H_2(s_j,p_j)^* \\
&+ (2\alpha p_i - s_i)\overline{(2\alpha p_j - s_j)})G(s_i,p_i)^*G(s_j,p_j) \\
= W_iW_j^* + 4H_1(s_i,p_i)H_1(s_j,p_j)^* &+ H_2(s_i,p_i)H_2(s_j,p_j)^* \\
&+ ((2-\alpha s_i)\overline{(2-\alpha s_j)})G(s_i,p_i)G(s_j,p_j)^*,
\end{aligned}
$$

which, as in the preceding section, indicates that there is a unitary V_1 from $\mathcal{L}_2 \oplus \mathcal{H} \oplus \mathcal{K}$ onto $\mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K}$ such that for all $v \in \mathcal{L}_2$,

$$
(4.3) \qquad \begin{pmatrix} I_{\mathcal{L}_2} \\ \mathcal{E}(s_j,p_j)^*H(s_j,p_j)^* \\ \overline{(2\alpha p_j - s_j)}G(s_j,p_j)^* \end{pmatrix} v \xrightarrow{V_1} \begin{pmatrix} f(s_j,p_j)^* \\ H(s_j,p_j)^* \\ \overline{(2-\alpha s_j)}G(s_j,p_j)^* \end{pmatrix} v
$$

where, as usual, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $H(s_j,p_j) = \big(2H_1(s_j,p_j), H_2(s_j,p_j)\big)$ and $\mathcal{E}(s_j,p_j)$ is the evaluation operator on $\mathcal{H} \oplus \mathcal{K}$. Now writing V_1 as the 3×3 block operator matrix in (3.12) and proceeding as before, we define

$$
f(t,q)^* = A_1(t,q) + B_1(t,q)\mathcal{E}(t,q)^*(I - D_1(t,q)\mathcal{E}(t,q)^*)^{-1}C_1(t,q),
$$

where the functions A_1, B_1, C_1 and D_1 are as in (3.17). This function interpolates the data because of (4.3). And it follows from Theorem 14 that the function f is in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$.

$\square$

5. The Toeplitz-corona on the symmetrized bidisc

In this section we prove the following general result from which follows Theorem 6. The lurking isometry argument, used to prove the realization theorem, is invoked here to construct the required functions.

Theorem 16. *Let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ be Hilbert spaces. Suppose $\Phi \in H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\big)$ and $\Theta \in H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_3, \mathcal{L}_2)\big)$ are given any functions. Then the following are equivalent:*

(i) *There exists a function Ψ in the closed unit ball of $H^\infty\big(\mathbb{G}, \mathcal{B}(\mathcal{L}_3, \mathcal{L}_1)\big)$ such that*

$$
\Phi(s,p)\Psi(s,p) = \Theta(s,p)
$$

for all $(s,p) \in \mathbb{G}$;

(ii) *The function*

$$[\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^*] \oslash K\big((s,p),(t,q)\big)$$

is positive semi-definite on $\mathbb{G}$ for every $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel K;

(iii) *There exist an $\alpha \in \overline{\mathbb{D}}$ and positive semi-definite $\mathcal{B}(\mathcal{L}_2)$-valued kernels Θ, Λ and Δ on $\mathbb{G}$ such that*

$$\begin{aligned}\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^* &= (4-s\bar{t})\Theta\big((s,p),(t,q)\big) + (1-p\bar{q})\Lambda\big((s,p),(t,q)\big)\\ &+ \big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\big)\Delta\big((s,p),(t,q)\big).\end{aligned}$$

Proof. The equivalence of *(ii)* and *(iii)* follows from Theorem 11.

(i) $\Rightarrow$ *(ii)* : Suppose *(i)* holds. Now *(ii)* follows from an application of Lemma 13 to the function Ψ and from the following simple observation:

$$\begin{aligned}&[\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^*] \oslash K\big((s,p),(t,q)\big)\\ = \;&\Phi(s,p)\big(I_{\mathcal{L}_1} - \Psi(s,p)\Psi(t,q)^*\big)\Phi(t,q)^* \oslash K\big((s,p),(t,q)\big).\end{aligned}$$

(ii) $\Rightarrow$ *(i)* : Suppose *(ii)* holds. Then since *(ii)* is equivalent to *(iii)*, we would have an $\alpha \in \overline{\mathbb{D}}$ and positive semi-definite $\mathcal{B}(\mathcal{L}_2)$-valued kernels Θ, Λ and Δ on $\mathbb{G}$ such that

$$\begin{aligned}\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^* &= (4-s\bar{t})\Theta\big((s,p),(t,q)\big) + (1-p\bar{q})\Lambda\big((s,p),(t,q)\big)\\ &+ \big((2-\alpha s)\overline{(2-\alpha t)} - (2\alpha p - s)\overline{(2\alpha q - t)}\big)\Delta\big((s,p),(t,q)\big).\end{aligned}$$

By Theorem 8, there exist auxiliary Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{K}$ and analytic functions $H_1 : \mathbb{G} \to \mathcal{B}(\mathcal{L}_2, \mathcal{H}_1)$, $H_2 : \mathbb{G} \to \mathcal{B}(\mathcal{L}_2, \mathcal{H}_2)$ and $G : \mathbb{G} \to \mathcal{B}(\mathcal{L}_2, \mathcal{K})$ such that

$$\begin{aligned}&\Theta\big((s,p),(t,q)\big) = H_1(s,p)H_1(t,q)^*, \Lambda\big((s,p),(t,q)\big) = H_2(s,p)H_2(t,q)^* \text{ and}\\ &\Delta\big((s,p),(t,q)\big) = G(s,p)G(t,q)^*.\end{aligned}$$

Consequently

$$\begin{aligned}&\Phi(s,p)\Phi(t,q)^* + s\bar{t}H_1(s,p)H_1(t,q)^* + p\bar{q}H_2(s,p)H_2(t,q)^* + (2\alpha p - s)\overline{(2\alpha q - t)}G(s,p)G(t,q)^*\\ = \;&\Theta(s,p)\Theta(t,q)^* + 4H_1(s,p)H_1(t,q)^* + H_2(s,p)H_2(t,q)^* + (2-\alpha s)\overline{(2-\alpha t)}G(s,p)G(t,q)^*,\end{aligned}$$

which implies that the linear map V_1 defined by

$$\begin{pmatrix}\Phi(t,q)^*\\ \mathcal{E}(t,q)^*H(t,q)^*\\ \overline{(2\alpha q - t)}G(t,q)^*\end{pmatrix} v \xrightarrow{V_1} \begin{pmatrix}\Theta(t,q)^*\\ H(t,q)^*\\ \overline{(2-\alpha t)}G(t,q)^*\end{pmatrix} v \tag{5.1}$$

for all $v \in \mathcal{L}_2$ is an isometry from a subspace of $\mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K}$ into $\mathcal{L}_3 \oplus \mathcal{H} \oplus \mathcal{K}$, where, as before, $H(s,p) = \big(2H_1(s,p), H_2(s,p)\big)$ is a function taking values in the space of operators from $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{L}_2$. We add an infinite-dimensional summand to $\mathcal{H}$ and $\mathcal{K}$, if necessary, to extend V_1 to a unitary from $\mathcal{L}_1 \oplus \mathcal{H} \oplus \mathcal{K}$ onto $\mathcal{L}_3 \oplus \mathcal{H} \oplus \mathcal{K}$. Decompose V_1 as the 3×3 block operator matrix in (3.12) and define the desired function Ψ by

$$\Psi(t,q)^* = A_1(t,q) + B_1(t,q)\mathcal{E}(t,q)^*(I - D_1(t,q)\mathcal{E}(t,q)^*)^{-1}C_1(t,q),$$

where the functions A_1, B_1, C_1 and D_1 are as in (3.17). Then Ψ is a contractive multiplier and by (5.1) it satisfies $\Psi(t,q)^*\Phi(t,q)^* = \Theta(t,q)^*$ for all $(t,q) \in \mathbb{G}$. Hence *(i)* holds. $\square$

Now Theorem 6 follows from Theorem 16, when we choose $\mathcal{L}_1 = \mathbb{C}^N$, $\mathcal{L}_2 = \mathbb{C} = \mathcal{L}_3$, $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_N)$ and Θ to be the constant function $\sqrt{\delta}$.

6. Acknowledgement

The author would like to thank Prof. John E. McCarthy profusely for clarifying a doubt which is used in the proof of Theorem 11 and Prof. T. Bhattacharyya for some helpful discussions.

References

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, Operator Theory: Advances and Applications, 48 (1990), 47-66.

[2] J. Agler and J. McCarthy, *Pick interpolation and Hilbert function spaces.* Graduate Studies in Mathematics, 44. American Mathematical Society, Providence, RI, 2002.

[3] J. Agler and J. McCarthy, *Nevanlinna-Pick interpolation on the bidisk*, J. Reine Angew. Math. 506(1999), 191-204.

[4] J. Agler, N. J. Young, *A commutant lifting theorem for a domain in* $\mathbb{C}^2$ *and spectral interpolation*, J. Funct. Anal. 161(1999), 452-477.

[5] J. Agler and N. J. Young, *Operators having the symmetrized bidisc as a spectral set*, Proc. Edinburgh Math. Soc. (2) 43 (2000), 195-210.

[6] J. Agler and N. J. Young, *A model theory for* Γ*-contractions*, J. Operator Theory 49 (2003), 45-60.

[7] J. Agler and N. J. Young, *The hyperbolic geometry of the symmetrized bidisc*, J. Geom. Anal. 14 (2004), 375-403.

[8] H. Bercovici, C. Foias, and A. Tannenbaum, *A spectral commutant lifting theorem*, Trans. Am. Math. Soc. 325 (1991), 741-763.

[9] T. Bhattacharyya, S. Pal and S. Shyam Roy, *Dilations of* Γ*-contractions by solving operator equations*, Advances in Mathematics, Volume 230, 2012, 577-606.

[10] T. Bhattacharyya and H. Sau, *Holomorphic functions on the symmetrized bidisk - realization, interpolation and Extension*, submitted, available at http://arxiv.org/pdf/1511.08962.pdf

[11] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. 76(1962), 547-559.

[12] J. B. Conway, *A course in functional analysis*, Second edition, Graduate texts in Mathematics; 96, Springer-Verlag New York, 1990.

[13] L. Hormander, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. 73(1967), 943-949.

[14] G. Misra, S. Shyam Roy and G. Zhang, *Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydisc*, Proc. Amer. Math. Soc., 141 (2013), 2361-2370.

[15] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann., 77(1916), 7-23.

[16] M. Rosenblum, *A coronal theorem for countably many functions* Integral Equations Operator Theory 3(1980), 125-137.

[17] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1991.

[18] J. Sarkar, Operator theory on symmetrized bidisc, *Indiana Univ. Math. J.* 64(2015), 847-873.

[19] C. F. Schubert, *The corona theorem as an operator theorem*, Proc. Amer. Math. Soc. 69(1978), 73-76.

[20] B. Sz.-Nagy, *A problem on operator valued bounded analytic functions*, Zap. Nauchn. Sem. LOMI 81(1978), 99.

[21] V. A. Tolokonnikov, *Estimates in Carlesons corona theorem and finitely generated ideals of the algebra* H^∞, Funktsional. Anal. i Prilozhen. 14 (1980), no. 4, 85-86 (Russian).

[22] S. Treil and B. D. Wick, *Analytic projections, coronal problem and geometry of holomorphic vector bundles*, J. Amer. Math. Soc. 22(2009), 55-76.

[23] *The corona problem. Connections between operator theory, function theory, and geometry*, Fields Institute Communications, 72. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2014.

Department of Mathematics, Indian Institute of Science, Bangalore-560012, India.
E-mail address: sau10@math.iisc.ernet.in